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Extremal Problems for Nonvanishing Functions in Bergman Spaces

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Dedicated to the memory of Semeon Yakovlevich Khavinson.

Abstract. In this paper, we study general extremal problems for non-vanishing functions in Bergman spaces. We show the existence and uniqueness of solutions to a wide class of such problems. In addition, we prove certain regularity results: the extremal functions in the problems considered must be in a Hardy space, and in fact must be bounded. We conjecture what the exact form of the extremal function is. Finally, we discuss the specific problem of minimizing the norm of non-vanishing Bergman functions whose first two Taylor coe cients are given.

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1. Introduction

For $0 < p < \infty$, let

$$A^p = \{f \text{ analytic in } \mathbb{D} : (\int |f(z)|^p dA(z))^{\frac{1}{p}} := \|f\|_{A^p} < \infty\}$$

denote the Bergman spaces of analytic functions in the unit disk $\mathbb{D}$. Here dA stands for normalized area measure $\frac{1}{\pi}dxdy$ in $\mathbb{D}$, $z = x + iy$. For $1 \leq p < \infty$, A^p is a Banach space with norm $\| \ \|_{A^p}$. A^p spaces extend the well-studied scale of Hardy spaces

$$H^p := \{f \text{ analytic in } \mathbb{D} : (\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi})^{\frac{1}{p}} := \|f\|_{H^p} < \infty\}.$$

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For basic accounts of Hardy spaces, the reader should consult the well-known monographs [Du, Ga, Ho, Ko, Pr]. In recent years, tremendous progress has been achieved in the study of Bergman spaces following the footprints of the Hardy spaces theory. This progress is recorded in two recent monographs [HKZ, DS] on the subject.

In H^p spaces, the theory of general extremal problems has achieved a state of finesse and elegance since the seminal works of S.Ya. Khavinson, and Rogosinski and Shapiro (see [Kh1, RS]) introduced methods of functional analysis. A more or less current account of the state of the theory is contained in the monograph [Kh2]. However, the theory of extremal problems in Bergman spaces is still at a very beginning. The main di culty lies in the fact that the Hahn-Banach duality that worked such magic for Hardy spaces faces tremendous technical di culty in the context of Bergman spaces because of the subtlety of the annihilator of the A^p space ($p \geq 1$) inside $L^p(dA)$. [KS] contains the first more or less systematic study of general linear extremal problems based on duality and powerful methods from the theory of nonlinear degenerate elliptic PDEs. One has to acknowledge, however, the pioneering work of V. Ryabych [Ry1, Ry2] in the 60s in which the first regularity results for solutions of extremal problems were obtained. Vukotić's survey ([Vu]) is a nice introduction to the basics of linear extremal problems in Bergman space. In [KS], the authors considered the problem of finding, for $1 < p < \infty$,

$$\sup\{|\int \bar{w} f \, dA| : \|f\|_{A^p} \leq 1\}, \tag{1.1}$$

where w is a given rational function with poles outside of $\mathbb{D}$. They obtained a structural formula for the solution (which is easily seen to be unique) similar to that of the Hardy space counterpart of problem (1.1). Note here that by more or less standard functional analysis, problem (1.1) is equivalent to

$$\inf\{\|f\|_{A^p} : f \in A^p, l_i(f) = c_i, i = 1, \dots, n\}, \tag{1.2}$$

where the $l_i \in (A^p)^*$ are given bounded linear functionals on A^p, $p > 1$. Normally, for l_i one takes point evaluations at fixed points of $\mathbb{D}$, evaluations of derivatives, etc... More details on the general relationship between problems (1.1) and (1.2) can be found in [Kh2, pp. 69-74]. For a related discussion in the Bergman spaces context, we refer to [KS, p. 960]. In this paper, we focus our study on problem (1.2) for nonvanishing functions. The latter condition makes the problem highly nonlinear and, accordingly, the duality approach does not work. Yet, in the Hardy spaces context, in view of the parametric representation of functions via their boundary values, one has the advantage of reducing the nonlinear problem for nonvanishing functions to the linear problem for their logarithms. This allows one to obtain the general structural formulas for the solutions to problems (1.1) or (1.2) for nonvanishing functions in Hardy spaces as well. We refer the reader to the corresponding sections in [Kh2] and the references cited there. Also, some of the specific simpler problems for nonvanishing H^p functions have recently been solved in [BK]. However, all the above-mentioned methods fail miserably in the context of Bergman spaces for the simple reason that there are no non-trivial Bergman functions that, acting as multiplication operators on Bergman spaces, are isometric.

Let us briefly discuss the contents of the paper. In Section 2, we study problem (1.2) for nonvanishing Bergman functions: we show the existence and uniqueness of the solutions to a wide class of such problems. Our main results are presented in Sections 2 and 3 and concern the regularity of the solutions: we show that although posed initially in A^p, the solution must belong to the Hardy space H^p, and hence, as in the corresponding problems in Hardy spaces in [Kh2], must be a product of an outer function and a singular inner function. Further, we show that that the

Proof. (The following argument is well known and is included for completeness.) Pick a sequence f_k of zero-free functions in A^p such that $l_i(f_k) = c_i$ for every $1 \leq i \leq n$ and every $k = 1, 2, \ldots$

Notice that by the same argument, the converse also holds; in other words, if we can solve the extremal problem in A^p for some $p > 0$, then we can also solve the extremal problem in A^2. Therefore for the remainder of the paper, we will consider only the case $p = 2$. Notice that if we consider Problem (1.2) without the restriction that f must be zero-free, the solution is very simple and well known. Considering for simplicity the case of distinct β_j, the unique solution is the unique linear combination of the reproducing kernels $k(., \beta_j)$ satisfying the interpolating conditions, where

$$k(z, w) := 1/(1 - \bar{w}z)^2.$$

Since our functions are zero-free, we will rewrite a function f as $f(z) =$ exp(ϕ(z

The next three lemmas are the technica

Therefore

$$F'(0) = \int |\exp(p_m^*(z))|^2 2Re(\prod_{i=1}^{n}(z-\beta_i)\psi_{m-n}(z))dA(z) = 0.$$

Replacing ψ_{m-n} by $i\psi_{m-n}$ gives

$$\int |\exp(p_m^*(z))|^2 2Re(\prod_{i=1}^{n}(z-\beta_i)i\psi_{m-n}(z))dA(z) = 0,$$

and therefore

$$\int |\exp(p_m^*(z))|^2 \prod_{i=1}^{n}(z-\beta_i)\psi_{m-n}(z)dA(z) = 0$$

for every polynomial ψ_{m-n} of degree at most $m-n$. □

Lemma 2.7. *For each $m \geq n$, $e^{p_m^*} \in H^2$, and he e H^2 norm are bo nded.*

Proof. Write

$$p_m^*(z) = L(z) + h(z)q_{m-n}(z),$$

where $L(z)$ is the Lagrange polynomial taking value c_i at β_i (for $i = 1, \ldots, n$), $h(z) = \prod_{i=1}^{n}(z-\beta_i)$, and q_{m-n} is a polynomial of degree at most $m-n$. We then have

$$\begin{aligned}
\int |e^{p_m^*(e^{i\theta})}|^2 d\theta &= i\int |e^{p_m^*(z)}|^2 z d\overline{z} \\
&= 2\int \frac{\partial}{\partial z}(|e^{p_m^*(z)}|^2 z)dA(z) \qquad \text{(by Green's formula)} \\
&= \int |e^{p_m^*(z)}|^2 (p_m^{*'}(z)z + 1)dA(z).
\end{aligned}$$

We would like to show that this integral is bounded by $C\|e^{p_m^*(z)}\|_{A^2}^2$, where C is a constant independent of m. First notice that

$$zp_m^{*'}(z) = zL'(z) + zh'(z)q_{m-n}(z) + zh(z)q'_{m-n}(z).$$

Since $zq'_{m-n}(z)$ is a polynomial of degree at most $m-n$, Lemma 2.6 allows us to conclude that

$$\int |e^{p_m^*(z)}|^2 zh(z)q'_{m-n}(z)dA(z) = 0.$$

On the other hand, $zL'(z)$ is bounded and independent of m, and therefore

$$|\int |e^{p_m^*(z)}|^2 zL'(z)dA(z)| \leq C_1\|e^{p_m^*(z)}\|_{A^2}^2,$$

where C_1 is a constant independent of m. Therefore the crucial term is that involving $zh'(z)q_{m-n}(z)$. Write

$$q_{m-n}(z) = q_{m-n}(\beta_k) + (z-\beta \quad .$$

where q_{m-n-1} is a polynomial of degree at most $m-n-1$. Then

$$zh'(z)q_{m-n}(z) = z\{\sum_{k=1}^{n}[\prod_{i=1, i\neq k}^{n}(z-\beta_i)]\}\{q_{m-n}(\beta_k) + (z-\beta_k)q$$

3. Another approach to regularity

In the following, we present a very di erent approach to showing the a priori regularity of the extremal function. It was developed by D. Aharonov and H.S. Shapiro in 1972 and 1978 in two unpublished preprints ([AhSh1, AhSh2]) in connection with their study of the minimal area problem for univalent and locally univalent functions. See also [ASS1, ASS2].

Given n points $\beta_1, \ldots, \beta_n$ of $\mathbb{D}$, and complex numbers $c_1, \ldots, c_n$ recall that L

Moreover, g_s is certainly zero-free, and hence so is f_s if we can verify that the polynomial $L(f/g_s)$ has no zeros in $\mathbb{D}$.

for some constant M, thus

$$\int |f$$

Proof. First note that the arguments based on (3.7 and 3.8) leading to (*)

so

$$|a_s(z) - z| \leq 4|h_s(z) - \frac{1}{1-z}| \leq 4\sum_{n=1}^{\infty} |b_{n,s} - 1||z|^n. \tag{3.17}$$

But, from (3.15)

$$|b_{n,s} - 1| = |\frac{\sin nt_0}{nt_0} - 1|.$$

Since the function

$$\frac{(\sin x)/x - 1}{x^2}$$

is bounded for x real, we have for some constant N :

$$|\frac{\sin nt_0}{nt_0} - 1| \leq N(nt_0)^2 \leq N'n^2s^2$$

for small s, in view of (3.13), where N' is some new constant. Thus, finally, inserting this last estimate into (3.17),

$$|a_s(z) - z| \leq N''s^2B(z),$$

where

$$B(z) := \sum_{n=1}^{\infty} n^2|z|^n,$$

which is certainlycercer

4. A discussion of the conjectured form of extremal functions

In this section we provide certain evidence in support of our overall conjecture and draw out possible lines of attack that would hopefully lead to a rigorous proof in the future. Recall that the extremal function f^* in the problem (2.1):

$$\lambda = \inf\{\| \exp(\phi(z))\|_{A^2} : \phi(\beta_i$$

desired Lipschitz regularity of the extremal functions. Surprisingly, as we show at the end of the paper, even in the simplest examples of problems for non-vanishing functions in A^2, if the extremals have the form (1.3), they fail to be even continuous in the closed disk. This may be the first example of how some extremals in A^p and H^p di er qualitatively. Of course, the extremal functions for Problem 2.1 in the H

subspace $[S]$ of A^2 generated by S that vanish at the points $\beta_1, \beta_2, \ldots, \beta_n$. In particular, by (4.6), f^* is orthogonal to all functions $\frac{\partial}{\partial z}(H \prod_{j=1}^{n}(z-\beta_j)^2 Sg)$ for all polynomials g, i.e.,

$$0 = \int \bar{f^*} \frac{\partial}{\partial z}(H \prod_{j=1}^{n}(z-\beta_j)^2 Sg) dA. \tag{4.8}$$

Applying Green's formula to (4.8), we arrive at

$$0 = \int \bar{f^*} H \prod_{j=1}^{n}(z-\beta_j)^2 Sg d\bar{z} = \int \bar{F} H \prod_{j=1}^{n}(z-\beta_j)^2 g \frac{dz}{}$$

on any Carleson set $K \subset \mathbb{T}$, then (4.7) implies right away that f^* is orthogonal to all functions in A^2 vanishing at $\beta_1, \beta_2, \ldots, \beta_n$, and hence

$$f^* = \sum_{j=1}^{n} \frac{a_j}{(1-\beta_j z)^2}$$

is a linear combination of reproducing kernels. Thus, we have the corollary already observed in ([AhSh1, AhSh2]):

Corollary 4.2. *If f^* i cyclic in A^2, i m be a ra ional f nc ion of he form* (4.2).

(ii) On the other hand, if we could a priori conclude that the singular part S of f^* is atomic (with spectral measure consisting of at most $2n-2$ atoms), then instead of using Carleson's theorem, we could simply take for the outer function H a polynomial $p \neq 0$ in $\mathbb{D}$ vanishing with multiplicity 2 at the atoms of S. Then following the above argument, once again we arrive at the conjectured form (1.3) for the extremal f^*.

Now, following S.Ya. Khavinson's approach to the problem (2.1) in the Hardy space context (see [Kh2, pp. 88]), we will sketch an argument, which perhaps, after some refinement, would allow us to establish the atomic structure of the inner factor S, using only the a priori H^2 regularity.

For that, define subsets B_r of spheres of radius r in A^2 :

$$B_r := \{f = e^{\phi} : \|f\|_{A^2} \leq r\},$$

where

$$\phi(z) = \quad 1$$

is the Poisson integral of ν, into $\mathbb{C}^n$ by

$$\Lambda(\nu) = (S(\nu)(\beta_j))_{j=1}^{n}.$$

Here

$$S(\nu)(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\nu(\theta) \tag{4.19}$$

stands for the Schwarz integral of the measure ν. Let us denote the image $\Lambda(\Sigma_r)$ in $\mathbb{C}^n$ by A_r

Indeed, if $R(e^{i\theta})$ (which is continuous on $\mathbb{T}$) were strictly negative on a subarc $E \subset \mathbb{T}$, by choosing $d\nu = sd\theta$ with s negative and arbitrarily large in absolute value on E and fixed on $\mathbb{T} - E$, we would make the left-hand side of (4.21) go to $+\infty$ while still keeping the constraints (4.16), (4.17) and (4.18) intact, thus violating (4.21).

The conjecture is intuitive in the sense that in order to maximize the integral in (4.21), we are best o if we concentrate all the negative contributions from the singular part of ν at the points where $R > 0$ is smallest. Note that this conjecture does correspond to the upper estimate of the number of atoms in the singular inner part of the extremal function f^* in (1.3). Indeed, R is a rational function of degree $2n$ and hence has $4n - 2$ critical points (i.e., where $R'(z) = 0$) in $\hat{\mathbb{C}}$. Since is initially stated as that of finding

$$\inf\{\int |F'(z)|^2 dA : F(0) = 0, F'(0) = 1, F''(0) = b, F'(z) \neq 0 \text{ in } \mathbb{D}\}. \tag{5.1}$$

Problem (5.1) has the obvious geometric meaning of finding, among all locally univalent functions whose first three Taylor coe cients are fixed, the one that maps the unit disk onto a Riemann surface of minimal area. Setting $f = F'$ and $c = 2b$ immediately reduces the problem to a particular example of problems mentioned in (4.23), namely that of finding

$$\inf\{\int |f|^2 dA : f \neq 0 \text{ in } \mathbb{D}, f(0) = 1, f'(0) = c\}. \tag{5.2}$$

Assuming without loss of generality that c is real, we find that the conjectured form of the extremal function f in (5.2) is

$$f(z) = C(z - A)e^{\mu_0 \frac{z+1}{z-1}}, \tag{5.3}$$

where $\mu_0 \geq 0$, and C, A, and μ_0 are uniquely determined by the interpolating conditions in (5.2). Of course, if $|c| \leq 1$ in (5.2), the obvious solution is

$$f^* = 1 + cz,$$

and hence, $F^* = z + \frac{c}{2}z^2$ solves (5.1), mapping $\mathbb{D}$ onto a cardioid. The nontrivial case is then when $|c| > 1$. All the results in the previous sections apply, so we know that the extremal for (5.2) has the form

$$f^* = hS,$$

where h is a bounded outer function and S is a singular inner function. As in Section 2, a simple variation gives us the orthogonality conditions (OC) as necessary conditions for extremality:

$$\int |f^*|^2 z^{n+2} dA = 0, \; n = 0, 1, 2, \ldots. \tag{5.4}$$

From now on, we will focus on the non trivial case of Problem 5.2 with $c > 1$. Thus, the singular inner factor of f^* is non trivial (cf. Corollary 4.2). In support of the conjectured extremal (5.3), we have the following proposition.

Proposition 5.1. *If he ing lar fac or* S *of* f^* *ha a ocia ed ing lar mea re* $d\mu$ *ha i a omic i h a ingle a om, hen*

$$f^*(z) = C(z - 1 - \mu_0)e^{\mu_0 \frac{z+1}{z-1}} \tag{5.5}$$

here C *and he eigh* μ_0 *are niq ely de ermined by he in erpola ing condi ion .*

Remark. C

(iii) The only remaining obstacle in solving the extremal problem (5.2) is showing a priori that the singular inner factor of the extremal function is a one atom singular function. If one follows the outline given in Section 4, we easily find that for the problem (5.2), the function $R(e^{i\theta})$ in (4.22) becomes a rational function of degree 2, and since $R \geq 0$ on $\mathbb{T}$,

$$R(e^{i\theta}) = \text{const}\,\frac{(e^{i\theta} - a)(1 - \overline{a}e^{i\theta})}{e^{i\theta}} = \text{const}\,|e^{i\theta} - a|^2, \tag{5.10}$$

where $|a| \leq 1$. Thus, as we have seen in Section 4, we would be done if we could show that the one atom measure is the solution of the extremal problem

$$\max\{\int R(e^{i\theta})d\mu(\theta) : \mu \leq 0, \mu \perp d\theta\} \tag{5.11}$$

where μ satisfies the constraint

$$\int |h|^2|S_\mu|^2 dA \leq 1 \tag{5.12}$$

for a given outer function h and R is given by (5.10). (Recall that S_μ

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