

## JENSEN TYPE INEQUALITIES AND RADIAL NULL SETS

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**Abstract.** We extend Jensen's formula to obtain an upper estimate of  $\log |f(0)|$  for analytic functions in the unit disk  $\mathbf{D}$  that are subject to a growth restriction. Suppose we have a closed subset  $E$  of the unit circle and  $f$  in addition is continuous in the union of the open disk and  $E$ . We obtain a formula that gives an upper estimate of  $\log |f(0)|$  in terms of the values of  $f$  on  $E$  and the so-called  $k$ -entropy of  $E$ . When the set  $E$  is taken to be the whole unit circle, we get the classical Jensen's inequality. Our formula is then applied to the study of radial null sets. 2000 Mathematics Subject Classification: 30H05, 30E25, 46E15.

### 1 Growth Spaces

In what follows,  $k$  denotes an increasing twice differentiable function that maps  $[0; 1) \rightarrow [0; \infty)$ .

## 2 Two Problems

(A) Find good (upper and lower) estimates for the quantity

$$J(Z; k) = \sup \{ \log \int f(0) j : f \in UBA^{<k>} ; \int j_Z = 0 \}$$

where  $Z = \{a_n\} \subset \mathbb{D}$  is a given sequence.

(B) Find good estimates for

$$J(E; \nu; k) = \sup \{ \log \int f(0) j : f \in UBA^{<k>} \setminus C(\mathbb{D} \setminus E) ; \int j_E = \nu \}$$

where  $E \subset \mathbb{D}$  is a closed set and  $\nu$  is a non-negative continuous function on  $E$ :

Note that for  $k \leq 0$ ; ( $A^{<0>} = H^1$ ) both problems have exact solutions:

$$J(Z; 0) = \sum_n \log \frac{1}{|a_n|}$$

$$J(E; \nu; 0) = \int_E \log \nu(z) dm(z)$$

where  $dm$  is the normalized Lebesgue measure on  $\mathbb{D}$ : (Here, we assume  $k$

and the radial projection of  $S$  :

$$PrS = \left\{ \frac{z}{|z|} : z \in S \right\}$$

Then we have

$$J(Z; \rho) \leq \inf_{S \subset Z} \left[ \rho \log \rho \int_S f(\frac{z}{|z|}) + \log \int_S f(\frac{z}{|z|}) \right] + \rho \log^+ \rho + C_\rho$$

and

$$J(Z; \rho) \geq \inf_{S \subset Z} \left[ \rho \log \rho \int_S f(\frac{z}{|z|}) + \log \int_S f(\frac{z}{|z|}) \right] + C_\rho$$

where  $C_\rho > 0$  depends only on  $\rho$ ; and the infima are taken over all finite subsets  $S$  of  $Z$  :

**COROLLARY 3.1** For a sequence  $Z$  such that  $0$  is not in  $Z$ ; define

$$D^+(Z) = \inf_{\rho} \left[ \rho \log \rho \int_S f(\frac{z}{|z|}) + \log \int_S f(\frac{z}{|z|}) \right] + \rho$$

Then  $D^+(Z) < \infty$  is necessary and  $D^+(Z) < \infty$  is sufficient for  $Z$  to be an  $A^1$  zero set.

Note that for other spaces  $A^{<k>}$  such that  $k$  has faster than logarithmic growth, a similar description of zero sets is not known.

## 4 Problem (B) for $A^{<k>}$

**THEOREM 4.1**

$$J(E; \rho; k) \leq \max_{E} \left[ \rho \log \rho \int_S f(\frac{z}{|z|}) + \log \int_S f(\frac{z}{|z|}) \right] + (\log \rho) \frac{\rho}{1-\rho} + C_\rho + \left(\frac{L}{\rho}\right)^{\log_2 C} \text{Entr}_k(E)$$

where  $0 < \rho < 1$ ,  $0 < \rho < \frac{1}{2}$  are arbitrary,  $C$  is the constant in (3),  $L$  is an absolute constant, and  $\text{Entr}_k(E)$  is the  $k$ -entropy of  $E$ , defined as follows:

$$\text{Entr}_k(E) = \int_0^1 k(t) dt$$

where  $f_{I_n}$  are the complementary arcs of  $E$ :

Special cases: (1)  $E = \partial\mathbf{D}$ : Letting  $\rho \rightarrow 0^+$ ; we get

$$J(\partial\mathbf{D}; \rho; k) \cdot \int_{\partial\mathbf{D}} \log \rho^{-1} dm(z)$$

which is the classical Jensen's inequality (in fact, equality.)

(2) If  $0 < \rho < 1$  on  $E$  and  $\rho = \max_{z \in E} \rho(z)$ ; we obtain

$$J(E; \rho; k) \cdot (\log \rho) \frac{jEj}{1} + \left(\frac{L}{jEj}\right)^{\log_2 C} \text{Entr}_k(E):$$

Choosing  $\rho = jEj^{-2}$ ; we get

$$J(E; \rho; k) \cdot \frac{1}{2} (\log \rho) jEj + \left(\frac{2L}{jEj}\right)^{\log_2 C} \text{Entr}_k(E):$$

(3) If  $\rho = 1$  and  $\rho = \frac{1}{2}$ ; then

$$J(E; \rho; k) \cdot \int_E \log^+ \rho^{-1} dm(z) + (2L)^{\log_2 C} \text{Entr}_k(E):$$

Proof: Write

$$\partial\mathbf{D} \setminus E = \bigcup_n I_n$$

where the  $I_n$  are open disjoint arcs on the unit circle. Call  $a_n$  and  $b_n$  the endpoints of  $I_n$ : Let  $0 < \rho < \frac{1}{2}$ : Let  $\rho_n$  be the open arc of the circle inside the unit disk passing through  $a_n$  and  $b_n$  and forming an angle of  $\frac{1}{4}\rho$  (we will think of it as the normalized angle  $\rho$ ) with the arc  $I_n$ : Let  $\tilde{E} = \bigcup_n \rho_n$ :  $\tilde{E}$  forms the boundary of an open subset  $\tilde{D}$  of the unit disk containing the origin. For the proof, we construct three functions  $U_1$ ;  $U_2$ ; and  $V$  as follows.

*Step 1: Construction of  $U_1$  and  $U_2$ :*

Define

$$U_1(z) = \int_E \text{Re} \left( \frac{3+z}{3-jz} \right) dm(z):$$

$U_1$  is the harmonic measure of  $E$  with respect to  $\mathbf{D}$ :

#### **LEMMA 4.1**

$$\lim_{r \rightarrow 1^-} U_1(r^3) = \hat{A}_E(z) \text{ a.e. on } \partial\mathbf{D}$$

where  $\hat{A}_E$  is the characteristic function of  $E$ : In addition,  $U_1(z) \leq \rho$  for  $z \in \tilde{E}$ :





Assume  $Entr_k(E)$  is finite and define the following harmonic function

$$V(z) = \int_{\partial\mathbf{D}} \frac{z}{\cdot} d\mu$$

By relabeling  $L$ ; we get the statement of the lemma.  $\square$

*Step 3: Construction of  $H$  and application of the maximum principle.* Finally, let us define

$$H(z) = U_2(z) + (\log p) \frac{1}{1 - U_1(z)} + L$$



$$\lim_{r \downarrow 1} f(r^3)$$